Limit calculation

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Prove that

$$\lim_{n\to\infty}\frac{1}{n^n}\prod_{k=1}^{n}\frac{n\sqrt{n}+(n+1)\sqrt{k}}{\sqrt{n}+\sqrt{k}}=\frac{4}{e}.$$

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Since $P_n := \frac{1}{n^n}\prod_{k=1}^{n}\frac{n\sqrt{n}+(n+1)\sqrt{k}}{\sqrt{n}+\sqrt{k}} = \frac{1}{n^n}\prod_{k=1}^{n}\frac{n\left(\sqrt{n}+\left(1+\frac{1}{n}\right)\sqrt{k}\right)}{\sqrt{n}+\sqrt{k}} =$

$\prod_{k=1}^{n}\frac{\sqrt{n}+\sqrt{k}+\frac{\sqrt{k}}{n}}{\sqrt{n}+\sqrt{k}} = \prod_{k=1}^{n}\left(1+\frac{\sqrt{\frac{k}{n}}}{n\left(1+\sqrt{\frac{k}{n}}\right)}\right)$

then $\ln P_n = \sum_{k=1}^{n}\ln\left(1+\frac{\sqrt{\frac{k}{n}}}{n\left(1+\sqrt{\frac{k}{n}}\right)}\right)$. Noting that $t-\frac{t^2}{2}<\ln(1+t)<t,$

for $t\in(0,1)$ we obtain

$$\sum_{k=1}^{n}\left(\frac{1}{n}\frac{\sqrt{\frac{k}{n}}}{1+\sqrt{\frac{k}{n}}}-\frac{1}{2n^2}\frac{\frac{k}{n}}{\left(1+\sqrt{\frac{k}{n}}\right)^2}\right)<\ln P_n<\sum_{k=1}^{n}\frac{1}{n}\frac{\sqrt{\frac{k}{n}}}{1+\sqrt{\frac{k}{n}}}\iff$$

$$\sum_{k=1}^{n}\frac{1}{n}\frac{\sqrt{\frac{k}{n}}}{1+\sqrt{\frac{k}{n}}}-\sum_{k=1}^{n}\frac{1}{2n^2}\frac{\frac{k}{n}}{\left(1+\sqrt{\frac{k}{n}}\right)^2}<\ln P_n<\sum_{k=1}^{n}\frac{1}{n}\frac{\sqrt{\frac{k}{n}}}{1+\sqrt{\frac{k}{n}}}.$$

Since

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{\sqrt{\frac{k}{n}}}{1+\sqrt{\frac{k}{n}}}=\int_0^1\frac{\sqrt{x}}{1+\sqrt{x}}dx=\left[\begin{array}{c} t:=\sqrt{x}\\ dx=2tdt\end{array}\right]=2\int_0^1\frac{t^2}{1+t}dt=2\ln 2-1$$

and $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2n^2}\frac{\frac{k}{n}}{\left(1+\sqrt{\frac{k}{n}}\right)^2}=\lim_{n\to\infty}\frac{1}{2n}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\frac{\frac{k}{n}}{\left(1+\sqrt{\frac{k}{n}}\right)^2}=$

$0\cdot\int_0^1\frac{x}{(1+\sqrt{x})^2}dx=0$ then $\lim_{n\to\infty}\ln P_n=2\ln 2-1.$

Therefore, $\lim_{n\to\infty}P_n=e^{2\ln 2-1}=\frac{4}{e}.$